PROJECTIVE DIFFERENTIAL GEOMETRY OF A PLANE CURVE

by

Alice V. Swenson
A.B. Bethany College, June, 1926.

Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Master of Arts.

Approved by:

E B Stouffer
Instructor in Charge

C. H. Ashton
Head or Chairman of Dept.

Date: May, 1928

TABLE OF CONTENTS

I.

INTRODUCTION

The study of curves from the point of view of their projective differential properties is recent. Halphen (1844-1889) was one of the first investigators in this field of geometry. Wilczynski* has, during the present century, built up a projective differential geometry of plane curves as well as of many other geometrical figures.

Still more recent is the development of the method used by E. B. Stouffer. His method applied to plane curves is given in his paper, "Some Canonical Forms and Associated Canonical Expansions."** This method is quite similar to that of Sannia.*** Stouffer's method of attack obtains a complete and independent system of invariants and covariants from the canonical form by a process which involves only direct substitutions. The geometric interpretation of them is made by means of the canonical expansion associated with the canonical equation.

* Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces.

** Stouffer, Bulletin of the American Mathematical Society for May-June, 1928, pp. 290-296.

*** Sannia, Atti dei Lincei (5) Vol.31, 1° sem., pp. 450-454 and pp. 503-506; (5) Vol. 31, 2° sem., pp. 17-19 and pp. 432-434.

The work in this thesis is an extension of the study of plane curves given in Stouffer's paper with further investigations which have suggested themselves as the work progressed.

FUNDAMENTAL NOTIONS

In differential geometry we regard a plane curve as the locus of points satisfying a system of parametric equations of the form, $y_i = f_i(x)$, $(i = 1,2,3)$, where x is the parameter. The homogeneous coordinates of any point P on the curve are (y_1, y_2, y_3). As x assumes various values, P moves along the curve. A particular value of x gives a unique point. We shall assume that the derivatives of $f_i(x)$ at $P(y_1, y_2, y_3)$ exist to as high an order as is necessary.

Let us consider a linear homogeneous differential equation of the form

$$(1)\quad y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0,$$

where p_i are functions of x, and where differentiation is with respect to the independent variable x. If we substitute $y_i = f_i(x)$, $(i = 1,2,3)$ in (1) we have

$$f_i'''(x) + 3p_1 f_i''(x) + 3p_2 f_i'(x) + p_3 f_i(x) = 0,\quad (i = 1,2,3)$$

It follows that the coefficients p_i may be determined so that $y_i = f_i(x)$ form a fundamental set of solutions of (1) provided $|f_i''\ f_i'\ f_i| \neq 0$.

We have demonstrated

Theorem I. Associated with a given parametric representation of a plane curve is a unique differential equation of the third order.

The integration of (1) would involve three arbitrary constants. According to the fundamental theorem of the theory of linear differential equations, if the coefficients of a differential equation are regular in the neighborhood of a given value of x, say $\underline{a}$, then there always exists a solution Y of the differential equation in the form

$$(2)\quad Y = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2} + y''(a)\frac{(x-a)^3}{3} + \cdots,$$

where $y(a)$, $y'(a)$, and $y''(a)$ are arbitrary constants.

For $x=a$ we have from (1)

$$y'''(a) + 3p_1 y''(a) + 3p_2 y'(a) + p_3 y(a) = 0.$$

We can solve this equation for $y'''(a)$, and by successive differentiation and substitution obtain $y^{(4)}(a)$ etc., in terms of y, y', and y''. When we substitute these values in (2) and collect our terms we have a general solution of the form

$$(3)\quad Y = \phi_1(x)y(a) + \phi_2(x)y'(a) + \phi_3(x)y''(a).$$

If we assign values to $y(a)$, $y'(a)$, and $y''(a)$ we have three particular solutions, $y_i = f_i(x)$ in the form

$$(4)\quad \begin{aligned} f_1(x) &= \phi_1(x)c_{11} + \phi_2(x)c_{12} + \phi_3(x)c_{13} \\ f_2(x) &= \phi_1(x)c_{21} + \phi_2(x)c_{22} + \phi_3(x)c_{23} \\ f_3(x) &= \phi_1(x)c_{31} + \phi_2(x)c_{32} + \phi_3(x)c_{33}, \end{aligned}$$

which we may solve for $\phi_i(x)$, $(i = 1,2,3)$, provided that $|c_{ij}| \neq 0$, $(i,j = 1,2,3)$.

Any other three particular solutions may be written as

$$Y_1(x) = \phi_1(x)k_{11} + \phi_2(x)k_{12} + \phi_3(x)k_{13}$$
$$(5)\quad Y_2(x) = \phi_1(x)k_{21} + \phi_2(x)k_{22} + \phi_3(x)k_{23}$$
$$Y_3(x) = \phi_1(x)k_{31} + \phi_2(x)k_{32} + \phi_3(x)k_{33}.$$

When we have substituted in (5) the values of $\phi_i(x)$ obtained from (4) we may write

$$Y_1(x) = \ell_{11} f_1(x) + \ell_{12} f_2(x) + \ell_{13} f_3(x)$$
$$(6)\quad Y_2(x) = \ell_{21} f_1(x) + \ell_{22} f_2(x) + \ell_{23} f_3(x)$$
$$Y_3(x) = \ell_{31} f_1(x) + \ell_{32} f_2(x) + \ell_{33} f_3(x).$$

If $|\ell_{ij}| \neq 0$, $(i,j = 1,2,3)$, the relations (6) are a fundamental set of solutions expressed as a linear combination of three other solutions. $Y_i(x)$ may be regarded as a parametric representation of a curve associated with the differential equation.

We may now state

Theorem II. Associated with a given differential equation of the third order there is a system of plane curves all projectively related.

III.

Transformation of Linear Homogeneous Differential Equation. Relative Invariants and Relative Covariants.

We may now transform the dependent variable y_i of our parametric representation, and, therefore, the y of our differential equation, by a point transformation of the form $y_i = \lambda(x)\,\bar{y}_i$. The point $P(y_1, y_2, y_3)$ is transformed into $P(\lambda\bar{y}_1, \lambda\bar{y}_2, \lambda\bar{y}_3)$, but, as we are dealing with homogeneous coordinates, the factor λ does not change the point. Therefore this transformation preserves the curve, and it is evident that it preserves the form of the differential equation.

We may transform the independent variable x of our parametric representation, and, therefore, of our differential equation, by a transformation of the form $\xi = \xi(x)$, where $x = \ell(\xi)$. If we substitute in $y_i = f_i(x)$ we have $y_i = f_i[\ell(\xi)]$, a new parametric representation of our curve which as a whole remains unchanged. Again it is evident that the form of the equation remains the same.

Hence we shall consider transformations of the form

$$(7) \qquad y_i = \lambda\bar{y}_i, \qquad \xi = \xi(x),$$

applicable to our differential equation. We shall assume that these functions may be differentiated up to the order required.

Wilczynski* begins with a very general point transformation and proves that the transformations of form (7) are the most general transformations which convert every equation of the form (1) into another of the same form.

We have seen that associated with a given curve is a whole set of differential equations related by transformations (7), and that the new equations are of the same form but with new coefficients. Properties of the coefficients which are unchanged by the transformations are uniquely associated with the curve.

Definition. A function of the new coefficients and the new dependent variable and their derivatives, which is equal, except for a factor, to the same function of the original coefficients and the original dependent variable and their derivatives, is called a relative covariant.

Definition. A function of the new coefficients and their derivatives, which is equal, except for a factor, to the same function of the original coefficients and their derivatives, is called a relative invariant.

From the transformation $\xi = \xi(x)$ we find

* Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Chapter I.

$$(8)\quad \begin{aligned} y' &= \bar{y}'\xi', \\ y'' &= \bar{y}''(\xi')^2 + \bar{y}'\xi'', \\ y''' &= \bar{y}'''(\xi')^3 + 3\bar{y}''\xi''\xi' + \bar{y}'\xi''', \quad \left(\bar{y}' = \frac{dy}{d\xi},\ \ \xi' = \frac{d\xi}{dx}\right) \end{aligned}$$

so that (1) becomes

$$(9)\quad \bar{y}''' + 3\bar{p}_1\bar{y}'' + 3\bar{p}_2\bar{y}' + \bar{p}_3\bar{y} = 0,$$

where

$$(10)\quad \begin{aligned} \bar{p}_1 &= \frac{1}{(\xi')}\left[p_1 + \eta\right], \\ \bar{p}_2 &= \frac{1}{(\xi')^2}\left[p_2 + \eta p_1 + 1/3\,(\eta' + \eta^2)\right], \\ \bar{p}_3 &= \frac{1}{(\xi')^3}\,p_3, \qquad \left(\eta = \frac{\xi''}{\xi'}\right). \end{aligned}$$

If $\xi'' = 0$, i.e., $\eta = 0$, the coefficients $\bar{p}_i$ of (9) are the same as those of (1) except for a factor.

The transformation $\bar{y} = \lambda(\xi)\,\bar{\bar{y}}$ gives us

$$(11)\quad \begin{aligned} \bar{y}' &= \lambda\bar{\bar{y}}' + \lambda'\bar{\bar{y}}, \\ \bar{y}'' &= \lambda\bar{\bar{y}}'' + 2\lambda'\bar{\bar{y}}' + \lambda''\bar{\bar{y}}, \\ \bar{y}''' &= \lambda\bar{\bar{y}}''' + 3\lambda'\bar{\bar{y}}'' + 3\lambda''\bar{\bar{y}}' + \lambda'''\bar{\bar{y}}. \end{aligned}$$

Upon substituting in (9) we obtain the new equation

$$(12)\quad \bar{\bar{y}}''' + 3\bar{P}_1\bar{\bar{y}}'' + 3\bar{P}_2\bar{\bar{y}}' + \bar{P}_3\bar{\bar{y}} = 0,$$

whose coefficients $\bar{P}_i$ are expressed by the equations

$$(13)\quad \begin{aligned} \bar{P}_1 &= \frac{1}{\lambda}\left[\lambda' + \lambda\bar{p}_1\right], \\ \bar{P}_2 &= \frac{1}{\lambda}\left[\lambda'' + 2\lambda'\bar{p}_1 + \lambda\bar{p}_2\right], \\ \bar{P}_3 &= \frac{1}{\lambda}\left[\lambda''' + 3\lambda''\bar{p}_1 + 3\lambda'\bar{p}_2 + \lambda\bar{p}_3\right]. \end{aligned}$$

If $\lambda' = 0$ and $\xi'' = 0$, the coefficients $\bar{P}_i$ of (12) are the same as those of (1) except for a factor.

Equations (8), (10), (11), and (13) show that if the transformations (7) transform equation (1) into a canonical form which is preserved only if

$$(14) \qquad \xi'' = 0 \,, \qquad \lambda' = 0,$$

then the new variable $\bar{\bar{y}}$ and the new coefficients $\bar{P}_i$ and their derivatives are determined except for multiplication by certain factors.

We have seen that transformations (7) applied to equation~~s~~ (1) give another equation of the same form in the variables $\bar{\bar{y}}$ and ξ, whose coefficients are expressible in terms of the original coefficients p_i and derivatives of λ and ξ. Such a differential equation shall be said to be equivalent to (1).

If we consider two equations

$$(a) \qquad y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0,$$

$$(b) \qquad y''' + 3q_1 y'' + 3q_2 y' + q_3 y = 0$$

which are equivalent under transformations (7) we can express the q's of (b) in terms of the p's of (a) and derivatives of λ and ξ. Moreover we have the following canonical forms of (a) and (b)

$$(\bar{a}) \qquad \bar{\bar{y}}''' + 3\bar{P}_2 \bar{\bar{y}}'' + \bar{P}_3 \bar{\bar{y}} = 0,$$

$$(\bar{b}) \qquad \bar{\bar{y}}''' + 3\bar{Q}_2 \bar{\bar{y}}'' + \bar{Q}_3 \bar{\bar{y}} = 0.$$

As before, if we determine η for the $\xi = \xi(x)$ and $\bar{y} = \lambda \bar{\bar{y}}$ transformations we have expressions for the $\bar{P}$'s in terms of the p's and for the $\bar{Q}$'s in terms of the q 's, where the expressions in the p's are the same as the expressions in the q's.

It is readily seen that, by successive applications of transformations$_x$ (7) we can transform equation ($\bar{a}$) into (a), (a) into (b), (b) into ($\bar{b}$) and, therefore, ($\bar{a}$) into ($\bar{b}$). But the form ($\bar{a}$) is maintained only if $\lambda' = 0$ and $\eta = 0$, the coefficients being multiplied by a factor. Therefore the coefficients of ($\bar{a}$) can differ from those of ($\bar{b}$) by a factor and the transformed expressions for the $\bar{P}$'s and the $\bar{Q}$'s in terms of the p's and q's respectively, differ by a factor. Therefore the coefficients of ($\bar{a}$) and all their derivatives are canonical forms of relative invariants.

The same type of argument shows that $\bar{\bar{y}}$ and its derivatives are canonical forms of relative covariants expressible in terms of the y's and coefficients p_i of (a).

IV.

Canonical Expansion. Canonical Form for Differential Equation (1). General Form for Relative Invariants and Relative Covariants.

We assume that our differential equation has been transformed into a canonical form with variables $\bar{\bar{y}}$ and ξ and coefficients $\bar{P}_i$ and proceed to determine the corresponding canonical expansion in non-homogeneous coordinates. From this expansion we shall determine the geometrical properties of our curve.

Let $\bar{\bar{y}}_i$ ($i = 1,2,3$) be the coordinates of a regular point on the curve. Further, let this point be given by the value of the parameter $\xi = 0$. Then, according to the general theory of differential equations, the coordinates $\bar{\bar{Y}}_i$ of any point $\bar{\bar{Y}}$ on the curve in the neighborhood of the point $\bar{\bar{y}}_i$ may be expressed as a power series in the new variables of the form

$$(15) \qquad \bar{\bar{Y}} = \bar{\bar{y}}(0) + \bar{\bar{y}}'(0)\,\xi + \bar{\bar{y}}''(0)\,\frac{\xi^2}{\lfloor 2} + \bar{\bar{y}}'''(0)\,\frac{\xi^3}{\lfloor 3} + \text{-----}.$$

When we attempt to substitute (12) and its derivatives in (14) we find that the results will be more simple if we make the coefficient of $\bar{\bar{y}}''$ equal zero.

We can make $\bar{P}_1 = 0$ by choosing λ to satisfy the equation $\lambda' + \lambda p_1 = 0$.

Equation (12) becomes

$$(16) \qquad \bar{\bar{y}}''' + 3\,\bar{P}_2\,\bar{\bar{y}}' + \bar{P}_3\,\bar{\bar{y}} = 0.$$

Equations (10) and (12) show that (16) is maintained if (14) is satisfied.

From (13), with $\frac{\lambda'}{\lambda} = -\bar{p}_1$, we have the coefficients

$$\bar{P}_1 = 0$$

(17) $$\bar{P}_2 = \bar{p}_2 - \bar{p}_1^2 - \bar{p}_1' ,$$

$$\bar{P}_3 = \bar{p}_3 - 3\,\bar{p}_1\bar{p}_2 + 2\,\bar{p}_1^3 - \bar{p}_1''$$

Upon substituting equations (10) in (17) we have

$$\bar{P}_2 = \frac{1}{(\xi')^2}\left[P_2 - \frac{2}{3}\eta' + \frac{1}{3}\eta^2\right],$$

(18)

$$\bar{P}_3 = \frac{1}{(\xi')^3}\left[P_3 - 3\eta P_2 + 3\eta\eta' - \eta^3 - \eta''\right],$$

where

$$P_2 = p_2 - p_1^2 - p_1',$$

$$P_3 = p_3 - 3p_1 p_2 + 2\,p_1^3 - p_1''.$$

We therefore, assume that $\bar{P}_1 = 0$ is one of the conditions which we may impose in order to give us a canonical expansion.

We shall calculate the coefficients of ξ in (15) up to the tenth order. After successive differentiation of equation (16) and after substitution of these results in (15), we have

(19) $$\bar{\bar{Y}} = \bar{\bar{y}} + \bar{\bar{y}}'\xi + \bar{\bar{y}}''\frac{\xi^2}{\lfloor 2} - \left[3\,\bar{P}_2\,\bar{\bar{y}}' + \bar{P}_3\,\bar{\bar{y}}\right]\frac{\xi^3}{\lfloor 3}$$

$$- \left[3\bar{P}_2\,\bar{\bar{y}}'' + (3\bar{P}_2' + \bar{P}_3)\,\bar{\bar{y}}' - \bar{P}_3'\,\bar{\bar{y}}\right]\frac{\xi^4}{\lfloor 4}$$

$$- \Big[(6\bar{P}_2' + \bar{P}_3)\,\bar{\bar{y}}'' + (2\bar{P}_3' + 3\bar{P}_2'' - 9\bar{P}_2^2)\,\bar{\bar{y}}'$$

$$+ (\bar{P}_3'' - 3\bar{P}_2\bar{P}_3)\,\bar{\bar{y}} \Big] \frac{\xi^5}{\underline{|5}}$$

$$- \Big[(9\bar{P}_2'' + 3\bar{P}_3' - 9\bar{P}_2^2)\,\bar{\bar{y}}'' + (3\bar{P}_3'' + 3\bar{P}_2'''$$

$$- 6\bar{P}_2\bar{P}_3 - 36\bar{P}_2\bar{P}_2')\,\bar{\bar{y}}'$$

$$+ (\bar{P}_3''' - 9\bar{P}_2'\bar{P}_3 - 3\bar{P}_2\bar{P}_3' - \bar{P}_3^2)\,\bar{\bar{y}} \Big] \frac{\xi^6}{\underline{|6}}$$

$$- \Big[(6\bar{P}_3'' + 12\bar{P}_2''' - 6\bar{P}_2\bar{P}_3 - 54\bar{P}_2\bar{P}_2')\,\bar{\bar{y}}''$$

$$+ (4\bar{P}_3''' + 3\bar{P}_2^{(4)} + 27\bar{P}_2^3 - \bar{P}_3^2 - 15\bar{P}_2'\bar{P}_3$$

$$- 18\bar{P}_2\bar{P}_3' - 36(\bar{P}_2')^2 - 63\bar{P}_2\bar{P}_2'')\,\bar{\bar{y}}'$$

$$+ (\bar{P}_3^{(4)} + 9\bar{P}_2^2\bar{P}_3 - 5\bar{P}_3\bar{P}_3' - 18\bar{P}_2''\bar{P}_3$$

$$- 12\bar{P}_2'\bar{P}_3' - 3\bar{P}_2\bar{P}_3'')\,\bar{\bar{y}} \Big] \frac{\xi^7}{\underline{|7}}$$

$$- \Big[(10\bar{P}_3''' + 15\bar{P}_2^{(4)} + 27\bar{P}_2^3 - \bar{P}_3^2 - 21\bar{P}_2'\bar{P}_3$$

$$- 24\bar{P}_2\bar{P}_3' - 90(\bar{P}_2')^2 - 117\bar{P}_2\bar{P}_2'')\,\bar{\bar{y}}''$$

$$+ (3\bar{P}_2^{(5)} + 5\bar{P}_3^{(4)} + 27\bar{P}_2^2\bar{P}_3 + 243\bar{P}_2^2\bar{P}_2'$$

$$- 99\bar{P}_2\bar{P}_2''' - 135\bar{P}_2'\bar{P}_2'' - 39\bar{P}_2\bar{P}_3''$$

$$- 45\bar{P}_2'\bar{P}_3' - 33\bar{P}_2''\bar{P}_3 - 7\bar{P}_3\bar{P}_3')\,\bar{\bar{y}}'$$

$$+ \left(\bar{P}_3^{(5)} + 6 \bar{P}_2 \bar{P}_3^{2} + 9 \bar{P}_2^{2} \bar{P}_3' + 72 \bar{P}_2 \bar{P}_2' \bar{P}_3 \right.$$

$$- 3 \bar{P}_2 \bar{P}_3''' - 15 \bar{P}_2' \bar{P}_3'' - 30 \bar{P}_2'' \bar{P}_3'$$

$$\left. - 30 \bar{P}_2''' \bar{P}_3 - 11 \bar{P}_3 \bar{P}_3'' - 5(\bar{P}_3')^2 \right) \bar{\bar{y}} \quad \frac{\xi^8}{\underline{|8}}$$

$$- \Big[\left(15 \bar{P}_3^{(4)} + 18 \bar{P}_2^{(5)} + 27 \bar{P}_2^{2} \bar{P}_3 + 324 \bar{P}_2^{2} \bar{P}_2' \right.$$

$$- 216 \bar{P}_2 \bar{P}_2''' - 432 \bar{P}_2' \bar{P}_2'' - 63 \bar{P}_2 \bar{P}_3''$$

$$\left. - 90 \bar{P}_2' \bar{P}_3' - 54 \bar{P}_2'' \bar{P}_3 - 9 \bar{P}_3 \bar{P}_3' \right) \bar{\bar{y}}''$$

$$+ \left(3 \bar{P}_2^{(6)} + 6 \bar{P}_3^{(5)} + 594 \bar{P}_2^{2} \bar{P}_2'' + 756 \bar{P}_2 (\bar{P}_2')^2 \right.$$

$$+ 108 \bar{P}_2^{2} \bar{P}_3' + 189 \bar{P}_2 \bar{P}_2' \bar{P}_3 + 9 \bar{P}_2 \bar{P}_3^{2}$$

$$- 144 \bar{P}_2 \bar{P}_2^{(4)} - 234 \bar{P}_2' \bar{P}_2''' - 72 \bar{P}_2 \bar{P}_3'''$$

$$- 99 \bar{P}_2' \bar{P}_3'' - 108 \bar{P}_2'' \bar{P}_3' - 63 \bar{P}_2''' \bar{P}_3$$

$$- 18 \bar{P}_3 \bar{P}_3'' - 135 (\bar{P}_2'')^2 - 12 (\bar{P}_3')^2$$

$$\left. - 81 \bar{P}_2^{4} \right) \bar{\bar{y}}' + \left(\bar{P}_3^{(6)} + \bar{P}_3^{3} + 36 \bar{P}_2 \bar{P}_3 \bar{P}_3' \right.$$

$$+ 27 \bar{P}_2' \bar{P}_3^{2} + 9 \bar{P}_2^{2} \bar{P}_3'' + 90 \bar{P}_2 \bar{P}_2' \bar{P}_3'$$

$$+ 189 \bar{P}_2 \bar{P}_2'' \bar{P}_3 + 162 (\bar{P}_2')^2 \bar{P}_3$$

$$- 3 \bar{P}_2 \bar{P}_3^{(4)} - 18 \bar{P}_2' \bar{P}_3''' - 45 \bar{P}_2'' \bar{P}_3''$$

$$- 60 \bar{P}_2''' \bar{P}_3' - 45 \bar{P}_2^{(4)} \bar{P}_3 - 21 \bar{P}_3 \bar{P}_3'''$$

$$- 21 \; \overline{P}_3{}' \; \overline{P}_3{}'' - 27 \, \overline{P}_2{}^3 \, \overline{P}_3 \quad) \; \overline{\overline{y}} \Big] \frac{\xi^9}{\lfloor 9} + \ldots$$

Since we are assuming that we have a canonical form, $\overline{\overline{y}}_i$, $\overline{\overline{y}}_i{}'$, $\overline{\overline{y}}_i{}''$ are the homogeneous coordinates of three fixed points not on a straight line. Therefore the coordinates of any point in the plane may be expressed in the form

$$\overline{\overline{Y}} = x_1 \overline{\overline{y}}_i + x_2 \overline{\overline{y}}_i{}' + x_3 \overline{\overline{y}}_i{}''.$$

Then the coordinates of the point $\overline{\overline{Y}}$ may be taken to be (x_1, x_2, x_3). They may be obtained directly from equation (20), and will be expansions in powers of ξ.

We shall change to non-homogeneous coordinates and obtain the ratios $\zeta = \dfrac{x_2}{x_1}$, $\eta = \dfrac{x_3}{x_1}$ in powers of ξ

$$\zeta = \xi - 3 \, \overline{P}_2 \frac{\xi^3}{\lfloor 3} - \Big[3 \, \overline{P}_2{}' - 3 \, \overline{P}_3 \Big] \frac{\xi^4}{\lfloor 4}$$

$$+ \Big[9 \, \overline{P}_2{}^2 - 3 \, \overline{P}_2{}'' + 3 \, \overline{P}_3{}' \Big] \frac{\xi^5}{\lfloor 5} + \Big[36 \, \overline{P}_2 \overline{P}_2{}' - 72 \, \overline{P}_2 \, \overline{P}_3$$

$$- 3 \, \overline{P}_2{}''' + 3 \, \overline{P}_3{}'' \Big] \frac{\xi^6}{\lfloor 6} + \Big[63 \, \overline{P}_2 \overline{P}_2{}'' + 36 \, (\overline{P}_2{}')^2$$

$$- 108 \, \overline{P}_2 \, \overline{P}_3{}' - 153 \, \overline{P}_2{}' \, \overline{P}_3 - 27 \, \overline{P}_2{}^3 - 3 \, \overline{P}_2{}^{(4)}$$

$$+ 99 \, \overline{P}_3{}^2 + 3 \, \overline{P}_3{}''' \Big] \frac{\xi^7}{\lfloor 7} + \Big[99 \, \overline{P}_2 \overline{P}_2{}'''$$

$$+ 135\,\bar{P}_2{}'\,\bar{P}_2{}'' - 3\,\bar{P}_2{}^{(5)} + 3\,\bar{P}_3{}^{(4)} - 153\,\bar{P}_2\,\bar{P}_3{}''$$

$$-261\,\bar{P}_2{}'\,\bar{P}_3{}' - 279\,\bar{P}_2{}''\,\bar{P}_3 - 243\,\bar{P}_2{}^2\,\bar{P}_2{}'$$

$$+ 345\,\bar{P}_3\,\bar{P}_3{}' + 1053\,\bar{P}_2{}^2\,\bar{P}_3 \Big] \frac{\xi^8}{\lfloor 8}$$

$$+\Big[144\,\bar{P}_2\,\bar{P}_2{}^{(4)} + 234\,\bar{P}_2{}'\,\bar{P}_2{}''' - 3\,\bar{P}_2{}^{(6)} + 3\,\bar{P}_3{}^{(5)}$$

$$- 207\,\bar{P}_2\,\bar{P}_3{}''' - 414\,\bar{P}_2{}'\,\bar{P}_3{}'' - 540\,\bar{P}_2{}''\,\bar{P}_3{}'$$

$$- 459\,\bar{P}_2{}'''\,\bar{P}_3 - 594\,\bar{P}_2{}^2\,\bar{P}_2{}'' - 756\,\bar{P}_2\,(\bar{P}_2{}')^2$$

$$+549\,\bar{P}_3\,\bar{P}_3{}'' + 135\,(\bar{P}_2{}'')^2 + 345\,(\bar{P}_3{}')^2$$

$$+ 1863\,\bar{P}_2{}^2\,\bar{P}_3{}' + 6885\,\bar{P}_2\,\bar{P}_2{}'\,\bar{P}_3 - 6885\,\bar{P}_2\,\bar{P}_3{}^2$$

$$+ 81\,\bar{P}_2{}^4 \Big] \frac{\xi^9}{\lfloor 9} + \text{-----}$$

$$\eta = \frac{1}{2}\xi^2 - 3\,\bar{P}_2\,\frac{\xi^4}{\lfloor 4} - \Big[6\,\bar{P}_2{}' - 9\bar{P}_3\Big]\frac{\xi^5}{\lfloor 5}$$

$$+\Big[9\,\bar{P}_2{}^2 - 9\,\bar{P}_2{}'' + 12\,\bar{P}_3{}'\Big]\frac{\xi^6}{\lfloor 6} + \Big[54\,\bar{P}_2\bar{P}_2{}' - 162\,\bar{P}_2\bar{P}_3$$

$$- 12\,\bar{P}_2{}''' + 15\,\bar{P}_3{}''\Big]\frac{\xi^7}{\lfloor 7} + \Big[117\,\bar{P}_2\,\bar{P}_2{}'' + 90(\bar{P}_2{}')^2$$

$$- 270\,\bar{P}_2\,\bar{P}_3{}' - 567\,\bar{P}_2{}'\,\bar{P}_3 - 27\bar{P}_2{}^3 - 15\,\bar{P}_2{}^{(4)}$$

$$+ 477\,\bar{P}_3{}^2 + 18\,\bar{P}_3{}'''\Big]\frac{\xi^8}{\lfloor 8} + \Big[216\,\bar{P}_2\bar{P}_2{}'''$$

$$+ 432\,\bar{P}_s'\,\bar{P}_s'' - 18\,\bar{P}_s^{(5)} + 21\,\bar{P}_s^{(4)} - 423\,\bar{P}_s\,\bar{P}_s''$$

$$- 1098\,\bar{P}_s'\,\bar{P}_s' - 1350\,\bar{P}_s''\,\bar{P}_s - 324\,\bar{P}_s^2\,\bar{P}_s'$$

$$+ 1971\,\bar{P}_s\,\bar{P}_s' + 2187\,\bar{P}_s^2\,\bar{P}_s\Big]\frac{\xi^9}{\underline{|9}} + \ldots\ldots$$

By the elimination of ξ from those equations we obtain, for the desired expansion,

$$(20)\qquad \eta = \frac{1}{2}\xi^2 + \frac{3}{8}\bar{P}_s\,\xi^4 + \frac{1}{40}\left(3\,\bar{P}_s' - 2\,\bar{P}_s\right)\xi^5$$

$$+ \frac{1}{240}\left(135\,\bar{P}_s^2 + 3\,\bar{P}_s'' - 2\,\bar{P}_s'\right)\xi^6$$

$$+ \frac{1}{1680}\left(459\,\bar{P}_s\,\bar{P}_s' - 306\,\bar{P}_s\,\bar{P}_s + 3\,\bar{P}_s''' - 2\bar{P}_s''\right)\xi^7$$

$$+ \frac{1}{13440}\Big(711\,\bar{P}_s\,\bar{P}_s'' + 459\,(\bar{P}_s')^2 - 474\,\bar{P}_s\bar{P}_s'$$

$$- 621\,\bar{P}_s'\,\bar{P}_s + 14175\,\bar{P}_s^3 + 3\bar{P}_s^{(4)} + 210\,\bar{P}_s^2$$

$$- 2\,\bar{P}_s'''\Big)\xi^8 + \frac{1}{120960}\Big(1035\,\bar{P}_s\,\bar{P}_s'''$$

$$+ 1629\,\bar{P}_s'\,\bar{P}_s'' + 3\,\bar{P}_s^{5} - 2\bar{P}_s^{(4)} - 690\,\bar{P}_s'\,\bar{P}_s''$$

$$- 1095\,\bar{P}_s'\,\bar{P}_s' - 1125\,\bar{P}_s''\,\bar{P}_s + 102303\,\bar{P}_s^2\,\bar{P}_s'$$

$$+ 756\,\bar{P}_s\,\bar{P}_s' - 63202\,\bar{P}_s^2\,\bar{P}_s\Big)\xi^9 + \ldots\ldots$$

The coefficient of ξ^5 in (20) is $(3\bar P_2' - 2\bar P_3)$. It is readily seen from equations (18) that

(21) $\quad \bar\theta_3 = 3\bar P_2' - 2\bar P_3 = \dfrac{1}{(\xi')^3}\left[3P_2' - 2P_3\right] = \dfrac{1}{(\xi')^3}\,\theta_3$

If θ_3 does not vanish identically we can make $\bar\theta_3 = 1$ by choosing $(\xi')^3 = \theta_3$. The conditions $\bar\theta_3 = 1$ and $\bar P_1 = 0$, characterizing our canonical form, are preserved only if $\xi'' = 0$, $\lambda' = 0$. Then our canonical form for (1) is

(22) $\quad \bar{\bar y}''' + 3\bar P_2 \bar{\bar y}' + \bar P_3 \bar{\bar y} = 0$, where $\bar\theta_3 = 1$ and our canonical expansion is

(23)
$$\eta = \frac{1}{2}\xi^2 + \frac{3}{8}\bar P_2\,\xi^4 + \frac{1}{40}\xi^5 + \frac{9}{16}\bar P_2^{\,2}\xi^6 + \frac{61}{560}\bar P_2\,\xi^7 + \frac{1}{4480}\left(17 - \bar P_3 + 4725\,\bar P_2^{\,3}\right)\xi^8 + \frac{1}{40320}\left(11367\,\bar P_2^{\,2} - \bar P_3'\right)\xi^9 + \cdots\cdots$$

We have shown that $\bar P_2$ and $\bar P_3$ and their derivatives are relative invariants in canonical form. If we substitute the value of $\eta = \xi''/\xi'$, obtained from $(\xi')^3 = \theta_3$, into equations (18) we obtain their general form

$$\bar P_2 = \frac{1}{\theta_3^{2/3}}\left[P_2 - 2/9\,\frac{\theta_3''}{\theta_3} + 7/27\left(\frac{\theta_3'}{\theta_3}\right)^2\right],$$

(24)
$$\bar P_3 = \frac{1}{\theta_3}\left[P_3 - P_2\frac{\theta_3'}{\theta_3} - 2\;8/27\left(\theta_3'/\theta_3\right)^3 + 4/3\;\theta_3'\theta_3''/\theta_3^{2} - 1/3\;\theta_3'''/\theta_3\right].$$

It is evident that $\bar{P}_2$ $\bar{P}_3$, and their derivatives, together with $\bar{\theta}_3$, form a complete system of invariants.

By means of equations (8), (10), (11), and (13) with the now known values of ξ'' and λ' i. e., $\eta =$ $1/3 \; \theta_3'/\theta_3$, we have the following general forms for the relative covariants.

$$
\begin{aligned}
(25) \qquad \bar{\bar{y}} &= 1/\lambda \, y, \\
\bar{\bar{y}}' &= \frac{1}{\lambda \xi'} \left[y' + \left(p_1 + \frac{1}{3} \frac{\theta_3'}{\theta_3}\right) y \right], \\
\bar{\bar{y}}'' &= \frac{1}{\lambda (\xi')^2} \left[y'' + \left(2p_1 + \frac{1}{3} \frac{\theta_3'}{\theta_3}\right) y' \right. \\
&\quad \left. + \left(p_1^2 + p_1' + 1/3 \, p_1 \frac{\theta_3'}{\theta_3} + 1/3 \, \frac{\theta_3 \theta_3'' - (\theta_3')^2}{\theta_3^2}\right) y \right].
\end{aligned}
$$

These covariants form a complete system of covariants, for the higher derivatives of $\bar{\bar{y}}$ than the second can be replaced by means of equation (22).

We now have a complete system of invariants and covariants which we have obtained from our canonical form by a process which involved only direct substitutions.

V.

Geometrical Determination of Triangle of Reference.

Now that we have our equation in an invariant form, we can investigate it geometrically. Associated with our curve there is a triangle of reference whose vertices are determined by the covariants $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$. We can locate these vertices geometrically by means of our canonical expansion. We shall determine the equations of curves which osculate our given curve at the point $\bar{\bar{y}}$.

The most simple osculating curve is the straight line which is tangent to the given curve at $\bar{\bar{y}}$ having first order contact, or two points of contact, at $\bar{\bar{y}}$. The point y is located on this tangent line whose equation is $x_3 = 0$ in homogeneous coordinates.

As the point $\bar{\bar{y}}$ generates the curve at $\bar{\bar{y}}$, $\bar{\bar{y}}'$ generates the curve at $\bar{\bar{y}}'$. The vertex $\bar{\bar{y}}''$ is somewhere on the tangent to the curve generated by $\bar{\bar{y}}'$, and is on the line $x_2 = 0$. Thus $\bar{\bar{y}}''$ can be determined by the intersection of these two lines.

By the method of undetermined coefficients we find from equation (23) a conic whose expansion coincides with the canonical expansion of our curve up to, and including, terms of the fourth order. In other words, this conic has at $\bar{\bar{y}}$ a contact of the fourth order with our curve, or it has five consecutive points in common with it. It is therefore, the osculating conic. In homogeneous coordinates its equation is

$$(26) \quad x_2^2 - 2x_1 x_3 + 3\bar{P}_2 x_3^2 = 0.$$

The polar of any point (x_1', x_2', x_3')

with respect to this conic is

$$-x_3' x_1 + x_2' x_2 + (3\bar{P}_2 x_3' - x_1')x_3 = 0.$$

The points $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$ have as their coordinates (1,0,0), (0,1,0), (0,0,1), respectively. The polar of $\bar{\bar{y}}'$ or (0,1,0) is $x_2 = 0$, i.e., the line on $\bar{\bar{y}}\ \bar{\bar{y}}''$ is the polar of $\bar{\bar{y}}'$ with respect to the osculating conic.

The line $\bar{\bar{y}}\ \bar{\bar{y}}''$, which has now a known geometrical significance, intersects the osculating conic in $\bar{\bar{y}}$ and in another point whose coordinates are $(3\bar{P}_2, 0, 2)$.

As ξ changes, and $\bar{\bar{y}}$ moves along the curve generated by it, the line $\bar{\bar{y}}\ \bar{\bar{y}}''$ will envelop a curve. We proceed to determine the point at which $\bar{\bar{y}}\ \bar{\bar{y}}''$ touches this curve. To do this, we let ξ increase by $\delta\xi$, an infinitesimal. The line $\bar{\bar{y}}\ \bar{\bar{y}}''$ will assume the position $\bar{\bar{y}} + \bar{\bar{y}}'\delta\xi \quad \bar{\bar{y}}'' + \bar{\bar{y}}'''\delta\xi$. To find the point where the line determined by the points $\bar{\bar{y}} + \bar{\bar{y}}'\delta\xi$, $\bar{\bar{y}}'' + \bar{\bar{y}}'''\delta\xi$ meets the line determined by the points $\bar{\bar{y}}$, $\bar{\bar{y}}''$ we put

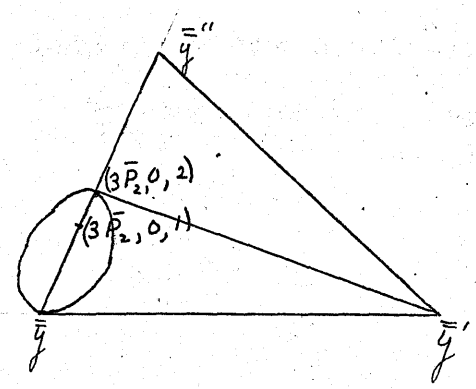
$$\alpha[\bar{\bar{y}} + \bar{\bar{y}}' d\xi + \dots] + \beta[\bar{\bar{y}} - (3\,\bar{P}_2\,\bar{\bar{y}}' + \bar{P}_3\,\bar{\bar{y}})\,d\xi + \dots].$$

Now, if $\alpha\, d\xi - 3\,\beta\,\bar{P}_2\, d\xi = 0$ the point is on the line $x_2 = 0$, therefore, $\alpha = 3\,\beta\,\bar{P}_2$, and we have for the point the expression $3\,\bar{P}_2\,[\bar{\bar{y}} + \dots] + \bar{\bar{y}}'' - \bar{P}_3\,\bar{\bar{y}}\,d\xi$. As $d\xi$ approaches zero, $3\,\bar{P}_2\,\bar{\bar{y}} + \bar{\bar{y}}''$ is a point on the line $x_2 = 0$. Its coordinates are $(3\bar{P}_2,\ 0,\ 1)$.

Now we have four points on the line $x_2 = 0$. These points have the coordinates $(0,0,1)$, $(3\bar{P}_2, 0, 2)$, $(3\bar{P}_2, 0, 1)$, $(1,0,0)$. The double ratio of these four points is -1, so they form a harmonic set. Thus the point $\bar{\bar{y}}''$ is uniquely determined. The line $x_1 = 3\,\bar{P}_2\,x_3$ is the polar line of the point $\bar{\bar{y}}''$.

The tangents to the conic at the points $(1,0,0)$, $(3\bar{P}_2, 0, 2)$ determine the pole $\bar{\bar{y}}'$ geometrically. Since the point $\bar{\bar{y}}'$ has the coordinates $(0,1,0)$, the equations of the tangents are $x_3 = 0$ and $3\,\bar{P}_2\,x_3 = 2\,x_1$.

Thus, by means of the osculating conic we have located the vertices $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$, and the sides $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, of our triangle of reference.



Again, by the method of undetermined coefficients we find from (23) equations of two of the eight pointic cubics whose expansions coincide with the canonical expansion of our curve up to, and including, terms of the seventh order. In other words, these cubics have at $\bar{\bar{y}}$ a contact of the seventh order with our curve, or they have eight consecutive points in common with it. They are, therefore, two osculating eight pointic cubics. In non-homogeneous coordinates their equations are

$$(27)\qquad u = 5\xi^3 + 15\,\bar{P}_2\,\xi\eta^2 + 2\eta^3 - 10\xi\eta = 0,$$

$$(28)\qquad v = 45\,\bar{P}_2\,\xi^2\eta - 7\xi\eta^2 + 135\,\bar{P}_2^{\,2}\eta^3 - 35\xi^2$$

$$- 195\,\bar{P}_2\,\eta^2 + 70\eta = 0.$$

In homogeneous coordinates these equations become

$$(29)\qquad u = 5x_2^3 + 15\,\bar{P}_2 x_2 x_3^2 + 2x_3^3 - 10\,x_1 x_2 x_3 = 0,$$

$$(30)\qquad v = 45\,\bar{P}_2\,x_2^2 x_3 - 7\,x_2 x_3^2 + 155\,\bar{P}_2^{\,2} x_3^3$$

$$- 35\,x_1 x_2^2 - 105\,\bar{P}_2\,x_1 x_3^2 + 70\,x_1^2 x_3 = 0.$$

Equation (29) is the equation of a cubic having a node at the point (1,0,0) because it satisfies the conditions that a plane curve has a node, namely $\frac{\partial f}{\partial x_1'} = 0$, $\frac{\partial f}{\partial x_2'} = 0$, $\frac{\partial f}{\partial x_3'} = 0$ where $(x_1', x_2', x_3') \equiv (1,0,0)$.

A nodal cubic has three points of inflexion. They are on the Hessian, a curve which meets the given curve only at the points of inflexion and multiple points of the curve. The three points of inflexion lie on a straight line.

The Hessian of our nodal cubic is

(31) $$15 x_2^3 + 6 x_3^3 + 10 x_1 x_2 x_3 - 15 \bar{P}_2 x_2 x_3^2 = 0$$

From the simultaneous solution of (31) and (29) we obtain

$$(3\bar{P}_2 , -2 (2/5)^{1/3}, 2)$$

$$(3\bar{P}_2 \; -2\omega (2/5)^{1/3}, 2)$$

$$(3\bar{P}_2 , -2\omega^2 (2/5)^{1/3}, 2) ,$$

as the points of inflexion. The equation of the line on these points is

(32) $3 \bar{P}_2 x_3 = 2x_1$

When we solve equation (29) with the equations $x_2 = ax_3$, $x_3 = bx_2$, general equations of lines through the point (1,0,0), we find that the lines $x_2 = 0$ and $x_3 = 0$ intersect and are tangent to the nodal cubic at the point (1,0,0).

Covariant Points in Special Positions.
Curve Generated by $\bar{\bar{y}}'$.

The vertices of our specialized triangle of reference are given in terms of the original coefficients and variables by

$$\bar{\bar{y}} = 1/_{\lambda}\, y,$$

(25)
$$\bar{\bar{y}}' = \frac{1}{\lambda \xi'}\left[y' + \left(p + 1/3\, \frac{\theta_3'}{\theta_3}\right) y\right],$$

$$\bar{\bar{y}}'' = \frac{1}{\lambda(\xi')^2}\left[y'' + \left(2p_1 + 1/3\, \theta_3'/\theta_3\right) y' + \left(p_1^2 + p_1' + 1/3\, p_1 \theta_3'/\theta_3 + 1/3\, \frac{\theta_3\theta_3'' - (\theta_3')^2}{\theta_3^2}\right) y\right]$$

The vertices of our general triangle of reference are y, y', y''. Since $\bar{\bar{y}}'$ is expressed as a linear combination of y and y', we have this side of the specialized triangle of reference identical with the corresponding side of the general triangle of reference, and $\bar{\bar{y}}'$ is a point on the line $y\ y'$.

The vertex $\bar{\bar{y}}$ is identical to the vertex y. Now if $p_1 = -1/3\, \theta_3'/\theta_3$, the vertex $\bar{\bar{y}}'$ is identical with y', and $\bar{\bar{y}}'' = \frac{1}{\lambda(\xi')^2}\left[y'' - 1/3\, \theta_3'/\theta_3\ y'\right]$, where, if θ_3 is a constant, the vertex $\bar{\bar{y}}''$ is identical with y''.

The equation of the curve generated by $\bar{\bar{y}}$ is

(22) $$\bar{\bar{y}}''' + 3\bar{P}_2\, \bar{\bar{y}}' + \bar{P}_3\, \bar{\bar{y}} = 0.$$

From this equation we obtain by differentiation and substitution the equations of the curves generated by $\bar{\bar{y}}'$, $\bar{\bar{y}}''$.

Thus the equation of the curve generated by $\bar{\bar{y}}$ is

(33) $$\bar{\bar{y}}^{(4)} - \frac{\bar{P}_3'}{\bar{P}_3}\,\bar{\bar{y}}''' + 3\,\bar{P}_2\,\bar{\bar{y}}'' + \left(3\,\bar{P}_2' + \bar{P}_3 - \frac{3\bar{P}_2\bar{P}_3'}{\bar{P}_3}\right)\bar{\bar{y}}' = 0.$$

If in (33) we put $\bar{\bar{y}}' = z$ we have

(34) $$z''' - \frac{\bar{P}_3'}{\bar{P}_3}\,z'' + 3\,\bar{P}_2\,z' + \left(3\bar{P}_2' + \bar{P}_3 - \frac{3\bar{P}_2\,\bar{P}_3'}{\bar{P}_3}\right) z = 0$$

which in form is similar to our original equation where

$$p_1 = -1/3\,\frac{\bar{P}_3'}{\bar{P}_3} \qquad p_2 = \bar{P}_2 \quad p_3 = 3\,\bar{P}_2' + \bar{P}_3 - \frac{3\bar{P}_2\bar{P}_3'}{\bar{P}_3}$$

From relations (25) we can write down immediately the co-variants

$$\bar{\bar{z}} = \frac{1}{\lambda}\,z,$$

(35) $$\bar{\bar{z}}' = \frac{1}{\lambda\xi'}\left[z' + \left(\frac{1}{3}\,\frac{\theta_3'}{\theta_3} - \frac{1}{3}\,\frac{\bar{P}_3'}{\bar{P}_3}\right) z\right],$$

$$\bar{\bar{z}}'' = \frac{1}{\lambda(\xi')^2}\Bigg[z'' + \left(\frac{1}{3}\,\theta_3'/\theta_3 - \frac{2}{3}\,\frac{\bar{P}_3'}{\bar{P}_3}\right) z'$$

$$+\Bigg(\frac{1}{9}\,\frac{(\bar{P}_3')^2}{\bar{P}_3^2} - \frac{1}{3}\,\frac{\bar{P}_3\,\bar{P}_3'' - (\bar{P}_3')^2}{\bar{P}_3^2} - \frac{1}{9}\,\frac{\theta_3'}{\theta_3}\,\frac{\bar{P}_3'}{\bar{P}_3}$$

$$+\frac{1}{3}\,\frac{\theta_3\,\theta_3'' - (\theta_3')^2}{\theta_3^2}\Bigg) z\Bigg].$$

Since

$$z = \bar{\bar{y}}'$$

We have

$$z' = \bar{\bar{y}}''$$

$$z'' = \bar{\bar{y}}'''$$

But $$\bar{\bar{y}}''' = -3\,\bar{P}_2\,\bar{\bar{y}}' - \bar{P}_3\,\bar{\bar{y}}$$

so that

$$z'' = -3\,\bar{P}_2\,\bar{\bar{y}}' - \bar{P}_3\,\bar{\bar{y}}.$$

Upon substitution in (55) we have

$$\bar{\bar{z}} = \frac{1}{\lambda}\,\bar{\bar{y}}',$$

$$\bar{\bar{z}}' = \frac{1}{\lambda\xi'}\left[\bar{\bar{y}}'' + \left(\frac{1}{3}\frac{\theta_3'}{\theta_3} - \frac{1}{3}\frac{\bar{P}_3'}{\bar{P}_3}\right)\bar{\bar{y}}'\right],$$

(56) $$\bar{\bar{z}}'' = \frac{1}{\lambda(\xi')^2}\left[\left(\frac{1}{3}\frac{\theta_3'}{\theta_3} - \frac{2}{3}\frac{\bar{P}_3'}{\bar{P}_3}\right)\bar{\bar{y}}'' + \left(\frac{1}{9}\frac{(\bar{P}_3')^2}{\bar{P}_3^2} - 1/3\,\frac{\bar{P}_3\,\bar{P}_3'' - (\bar{P}_3')^2}{\bar{P}_3^2} - 1/9\,\frac{\theta_3'}{\theta_3}\frac{\bar{P}_3'}{\bar{P}_3} + \frac{1}{3}\,\frac{\theta_3\theta_3'' - (\theta_3')^2}{\theta_3^2} - 3\,\bar{P}_2\right)\bar{\bar{y}}' - \bar{P}_3\,\bar{\bar{y}}\right].$$

By means of equations (24) and (25) we may express (36) in terms of the original coefficients and variables.

The triangle of reference for the curve generated by $\bar{\bar{y}}'$ with vertices $\bar{\bar{z}}$, $\bar{\bar{z}}'$, $\bar{\bar{z}}''$ is not, in general, the same as that for the curve generated by $\bar{\bar{y}}$.

The vertex $\bar{\bar{z}}$ is identical with $\bar{\bar{y}}'$, $\bar{\bar{z}}'$, is on the line $\bar{\bar{y}}'\ \bar{\bar{y}}''$, and $\bar{\bar{z}}''$ is a point in the plane.

If

$$\frac{\theta_3'}{\theta_3} = \frac{\bar{P}_s'}{\bar{P}_s}$$

$\bar{\bar{z}}'$ is identical with the vertex $\bar{\bar{y}}''$, and

$$\bar{\bar{z}}'' = \frac{1}{\lambda(\xi')^2}\left[-1/3\,\frac{\bar{P}_s'}{\bar{P}_s}\,\bar{\bar{y}}'' - 3\,\bar{P}_s\,\bar{\bar{y}}' - \bar{P}_s\,\bar{\bar{y}}\right].$$

Now if $\bar{P}_s' = 0$

$$\bar{\bar{z}}'' = \frac{1}{\lambda(\xi')^2}\left[-3\,\bar{P}_s\,\bar{\bar{y}}' - \bar{P}_s\,\bar{\bar{y}}\right]$$

where if $\bar{P}_s = 0$

$$\bar{\bar{z}}'' = \frac{1}{\lambda(\xi')^2}\left[-\bar{P}_s\,\bar{\bar{y}}\right]$$

so that $\bar{\bar{z}}''$ is identical with $\bar{\bar{y}}$.

The study of this curve generated by $\bar{\bar{y}}'$ can proceed as that for the curve generated by $\bar{\bar{y}}$ but the calculations become rather complicated and lengthy. The same is true for the curve generated by $\bar{\bar{y}}''$.

Covariant Points $\bar{\bar{y}}^{(4)}$ etc. Effect on Curves and Points When Invariants have Special Values.

By successive differentiation of (22) we obtain expressions for $\bar{\bar{y}}^{(4)}$ etc., as linear combinations of $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$ which means that they are points in *the* plane $\bar{\bar{y}}\ \bar{\bar{y}}'\ \bar{\bar{y}}''$. The point $\bar{\bar{y}}^{(3)}$ lies on the tangent to the curve at the vertex $\bar{\bar{y}}''$, $\bar{\bar{y}}^{(4)}$ lies on the tangent to the curve at the point $\bar{\bar{y}}^{(3)}$ etc. Let us investigate under what conditions these points will be in special positions.

From (22) we have

$$
\begin{aligned}
\bar{\bar{y}}^{(4)} &= -3\bar{P}_2\,\bar{\bar{y}}'' - \left[3\bar{P}_2' + \bar{P}_3\right]\bar{\bar{y}}' - \bar{P}_3'\,\bar{\bar{y}}, \\
\bar{\bar{y}}^{(5)} &= -\left[6\bar{P}_2' + \bar{P}_3\right]\bar{\bar{y}}'' + \left[9\bar{P}_2^2 - 3\bar{P}_2'' - 2\bar{P}_3'\right]\bar{\bar{y}}' \\
&\quad + \left[3\bar{P}_2\,\bar{P}_3 - \bar{P}_3''\right]\bar{\bar{y}}, \\
\bar{\bar{y}}^{(6)} &= \left[9\bar{P}_2^2 - 9\bar{P}_2'' - 3\bar{P}_3'\right]\bar{\bar{y}}'' + \left[36\bar{P}_2\bar{P}_2' + 6\bar{P}_2\bar{P}_3 - 3\bar{P}_2''' - 3\bar{P}_3''\right]\bar{\bar{y}}' \\
&\quad + \left[3\bar{P}_2\,\bar{P}_3' + 9\bar{P}_2'\,\bar{P}_3 + \bar{P}_3^2 - \bar{P}_3'''\right]\bar{\bar{y}},
\end{aligned}
\tag{37}
$$

If, in (22) we let $\bar{P}_2 = 0$ we have

$$\bar{\bar{y}}''' + \bar{P}_3\,\bar{\bar{y}} = 0$$

so that $\bar{\bar{y}}'''$ is the point $\bar{\bar{y}}$, and the curve at $\bar{\bar{y}}''$ is tangent to the line $\bar{\bar{y}}\ \bar{\bar{y}}''$.

The curve generated by $\bar{\bar{y}}'$ becomes

$$\bar{\bar{y}}^{(4)} + \bar{P}_3 \bar{\bar{y}}' + \bar{P}_3' \bar{\bar{y}} = 0$$

but

$$\bar{\bar{y}} = -\frac{1}{\bar{P}_3} \bar{\bar{y}}'''$$

and we have

$$\bar{\bar{y}}^{(4)} - \frac{\bar{P}_3'}{\bar{P}_3} \bar{\bar{y}}''' + \bar{P}_3 \bar{\bar{y}}' = 0$$

where $\bar{P}_3' = 0$, since $\bar{P}_3 = -1/2$.

Thus we have

$$\bar{\bar{y}}^{(4)} + \bar{P}_3 \bar{\bar{y}}' = 0$$

so that $\bar{\bar{y}}^{(4)}$ is the point $\bar{\bar{y}}'$.

From the last equation we obtain the curve generated by $\bar{\bar{y}}''$

$$\bar{\bar{y}}^{(5)} + \bar{P}_3 \bar{\bar{y}}'' + \bar{P}_3' \bar{\bar{y}}' = 0$$

but

$$\bar{\bar{y}}' = -1/\bar{P}_3 \; \bar{\bar{y}}^{(4)}$$

so we have

$$\bar{\bar{y}}^{(5)} - \frac{\bar{P}_3'}{\bar{P}_3} \bar{\bar{y}}^{(4)} + \bar{P}_3 \bar{\bar{y}}'' = 0$$

where again $\bar{P}_3' = 0$.

Thus we have $\bar{\bar{y}}^{(5)} + \bar{P}_3 \bar{\bar{y}}'' = 0$

and the point $\bar{\bar{y}}^{(5)}$ is the point $\bar{\bar{y}}''$. Proceeding thus, we find that the points line up in this manner

$$\bar{\bar{y}} \equiv \bar{\bar{y}}^{(3)} \equiv \bar{\bar{y}}^{(6)} ----$$

$$\bar{\bar{y}}' \equiv \bar{\bar{y}}^{(4)} \equiv \bar{\bar{y}}^{(7)} ----$$

$$\bar{\bar{y}}'' \equiv \bar{\bar{y}}^{(5)} \equiv \bar{\bar{y}}^{(8)} ----$$

Similarly, the curves at these points are identical.

When $\bar{P}_2 = 0$, $\bar{P}_3 = -1/2$ and $\bar{P}_3' = 0$, and our canonical expansion becomes

$$\eta = 1/2\,\xi^2 + 1/40\,\xi^5 + 1/256\,\xi^8 + 0.\,\xi^9 + \ldots$$

The osculating conic becomes

$$x_2^2 - 2x_1 x_3 = 0.$$

The osculating nodal cubic and the osculating nine-pointic cubic become

$$5x_2^3 + 2x_3^3 - 10\,x_1 x_2 x_3 = 0$$

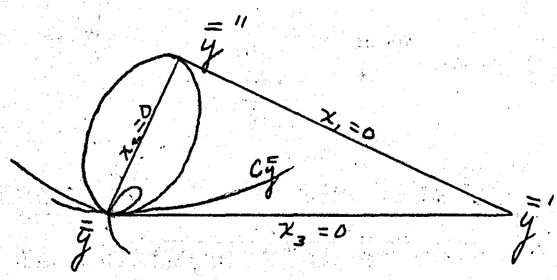
The Halphen point becomes the point $\bar{\bar{y}}$, and its polar line is $x_3 = 0$. Halphen* has called such points of a curve which coincide with their Halphen point, coincidence points.

* Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces. page 68.

The osculating conic is tangent to the line $x_1 = 0$ for the point $(3\bar{P}_2 \ 0,2)$ becomes the point $(0,0,1)$. The points of inflexion of our cubic lie on the line $x_1 = 0$.

This gives us the interpretation of what is known as the Laguerre-Forsyth canonical form where

$$\bar{P}_1 = 0, \quad \bar{P}_2 = 0 \text{ and } \bar{\theta}_3 = 1$$



If in (22) we let $\bar{P}_3 = 0$, we have

$$\bar{\bar{y}}''' + 3\bar{P}_2 \ \bar{\bar{y}}' = 0,$$

so that $\bar{\bar{y}}'''$ is the point $\bar{\bar{y}}'$. This equation is a differential equation of a straight line.

The curve generated by $\bar{\bar{y}}'$ is

$$\bar{\bar{y}}^{(4)} + 3\bar{P}_2 \ \bar{\bar{y}}'' + 3 \ \bar{P}_2' \ \bar{\bar{y}}' = 0$$

but

$$\bar{\bar{y}}' = -\frac{1}{3\,\bar{P}_2}\,\bar{\bar{y}}'''$$

so that

$$\bar{\bar{y}}^{(4)} - \frac{\bar{P}_2'}{\bar{P}_2}\,\bar{\bar{y}}''' + 3\,\bar{P}_2\,\bar{\bar{y}}'' = 0$$

where $\bar{P}_2' = 1/3$ since $\bar{P}_3 = 0$.

Thus, we have

$$\bar{\bar{y}}^{(4)} - \frac{1}{3\,\bar{P}_2}\,\bar{\bar{y}}''' + 3\,\bar{P}_2\,\bar{\bar{y}}'' = 0,$$

and the point $\bar{\bar{y}}^{(4)}$ is on the line $\bar{\bar{y}}''\ \bar{\bar{y}}'''$.

Again this is a differential equation of a straight line.

The curve generated by $\bar{\bar{y}}''$ is

$$\bar{\bar{y}}^{(5)} - 2\,\frac{\bar{P}_2'}{\bar{P}_2}\,\bar{\bar{y}}^{(4)} - \left[\frac{\bar{P}_2\bar{P}_2'' - 2\,(\bar{P}_2')^2}{\bar{P}_2^{\,2}} + 3\bar{P}_2\right]\bar{\bar{y}}''' = 0$$

where $\bar{P}_2'' = 0$ since $\bar{P}_2' = 1/3$ and we have

$$\bar{\bar{y}}^{(5)} - \frac{2}{3\,\bar{P}_2}\,\bar{\bar{y}}^{(4)} + \left[\frac{2}{9\,\bar{P}_2^{\,2}} - 3\,\bar{P}_2\right]\bar{\bar{y}}''' = 0.$$

If we continue in this manner we find that the condition $\bar{P}_3 = 0$ gives us differential equations of straight lines generated by the points $\bar{\bar{y}}'$ $\bar{\bar{y}}''$ etc. Further, all the points $y^{(k)}$, $(k = 3,4,5,6,\text{---}\ n)$, are on the line $\bar{\bar{y}}'\ \bar{\bar{y}}''$.

When $\bar{P}_3 = 0$, our canonical expansion becomes

$$\eta = \frac{1}{2}\xi^2 + 3/8\ \bar{P}_2 \xi^4 + 1/40\ \xi^5 + 9/16\ \bar{P}_2^2 \xi^6$$

$$+ \frac{51}{560} \bar{P}_2 \xi^7 + \frac{1}{4480}\left[17 + 4725\ \bar{P}_2^3\right] \xi^8$$

$$+ \frac{421}{4480} \bar{P}_2^2 \xi^9 + \cdots\cdots$$

The equations of the osculating conic and osculating nodal cubic remain the same. The equation of the osculating nine-pointic cubic becomes

$$1015\ x_2^3 - 2700\ \bar{P}_2^2\ x_2^2\ x_3 + 3465\ \bar{P}_2 x_2 x_3^2$$

$$+\left[406 - 8100\ \bar{P}_2^3\right]\ x_3^3 + 2100\ \bar{P}_2\ x_1 x_2^2$$

$$-2030\ x_1\ x_2\ x_3 + 11700\ \bar{P}_2^2\ x\ \ x_3^2$$

$$-4200\ \bar{P}_2\ x_1^2 x_3 = 0.$$

The Halphen point and its polar line are unchanged.

If we let

$$\bar{P}_2' = 0$$

then

$$\bar{P}_3 = -1/2\ ,\ \bar{P}_3' = 0\ .$$

and equations (37) become

$$\bar{\bar{y}}''' = -3\ \bar{P}_2\ \bar{\bar{y}}' - \bar{P}_3\ \bar{\bar{y}}\ ,$$

$$\bar{\bar{y}}^{(4)} = -3\ \bar{P}_2\ \bar{\bar{y}}'' + 1/2\ \bar{\bar{y}}'\ ,$$

$$\bar{\bar{y}}^{(5)} = 1/2\ \bar{\bar{y}}'' + 9\ \bar{P}_2^2\ \bar{\bar{y}}' - 3/2\ \bar{P}_2\ \bar{\bar{y}},$$

$$y^{(6)} = 9\,\bar{P}_s^{2}\,\bar{\bar{y}}'' - 3\,\bar{P}_s\,\bar{\bar{y}}' + 1/4\,\bar{\bar{y}}$$

The point $\bar{\bar{y}}^{(3)}$ is on the line $\bar{\bar{y}}\,\bar{\bar{y}}'$.

The point $\bar{\bar{y}}^{(4)}$ is on the line $\bar{\bar{y}}'\,\bar{\bar{y}}''$.

The points $\bar{\bar{y}}^{(5)}, \bar{\bar{y}}^{(6)}$ are in the plane.

If we let

$$\bar{P}_s' = 0$$

we obtain from (37)

$$\bar{\bar{y}}''' = -3\bar{P}_s\,\bar{\bar{y}}' - \bar{P}_s\,\bar{\bar{y}},$$

$$\bar{\bar{y}}^{(4)} = -3\,\bar{P}_s\,\bar{\bar{y}}'' - \left[1+3\bar{P}_s\right]\bar{\bar{y}}',$$

$$\bar{\bar{y}}^{(5)} = -\left[2+5\,\bar{P}_s\right]\bar{\bar{y}}'' + 9\bar{P}_s^{2}\,\bar{\bar{y}}' + 3\bar{P}_s\bar{P}_s\,\bar{\bar{y}},$$

$$\bar{\bar{y}}^{(6)} = 9\,\bar{P}_s^{2}\,\bar{\bar{y}}'' + \left[36\,\bar{P}_s\,\bar{P}_s' + 6\,\bar{P}_s\bar{P}_s\right]\bar{\bar{y}}'$$
$$+\left[9\,\bar{P}_s'\,\bar{P}_s + \bar{P}_s^{2}\right]\bar{\bar{y}}.$$

The point $\bar{\bar{y}}^{(3)}$ is on the line $\bar{\bar{y}}\;\bar{\bar{y}}'$

The point $\bar{\bar{y}}^{(4)}$ is on the line $\bar{\bar{y}}'\;\bar{\bar{y}}''$

The points $\bar{\bar{y}}^{(5)}, \bar{\bar{y}}^{(6)}$ are in the plane.

If we let

$$\left[3\bar{P}_s' + \bar{P}_s\right] = 0$$

Then $\bar{P}_s = -1/3 \quad \bar{P}_s' = 1/9$

so that equations (37) become

$$\bar{\bar{y}}''' = -3\,\bar{P}_2\,\bar{\bar{y}}' + 1/3\,\bar{\bar{y}},$$

$$\bar{\bar{y}}^{(4)} = -3\,\bar{P}_2\,\bar{\bar{y}}'',$$

$$\bar{\bar{y}}^{(5)} = -1/3\,\bar{\bar{y}}'' + 9\,\bar{P}_2^2\,\bar{\bar{y}}' - \bar{P}_2\,\bar{\bar{y}},$$

$$\bar{\bar{y}}^{(6)} = 9\,\bar{P}_2^2\,\bar{\bar{y}}'' + 2\bar{P}_2\,\bar{\bar{y}}' - 2/9\,\bar{\bar{y}}$$

The point $\bar{\bar{y}}'''$ is on the line $\bar{\bar{y}}\quad\bar{\bar{y}}'$.

The point $\bar{\bar{y}}^{(4)}$ is the point $\bar{\bar{y}}''$.

The points $\bar{\bar{y}}^{(5)}$, $\bar{\bar{y}}^{(6)}$ are in the plane.

The results of this section may be tabulated as follows:

If $\bar{P}_2 = 0$	The point $\bar{\bar{y}} \equiv \bar{\bar{y}}''' \equiv \bar{\bar{y}}^{(6)} \equiv$. ------ The point $\bar{\bar{y}}' \equiv \bar{\bar{y}}^{(4)} \equiv \bar{\bar{y}}^{(7)} \equiv$ ------ The point $\bar{\bar{y}}'' \equiv \bar{\bar{y}}^{(5)} \equiv \bar{\bar{y}}^{(8)} \equiv$ ------
If $\bar{P}_3 = 0$	The point $\bar{\bar{y}}''' \equiv \bar{\bar{y}}'$. The points $\bar{\bar{y}}^{(k)}$, $(k=4,5 \text{ --- } n)$ are points on the line $\bar{\bar{y}}' \ \bar{\bar{y}}''$. The curves generated by the points $\bar{\bar{y}}^{(k)}$, $(k=1,2 \text{ --- } n)$ are straight lines.
If $\bar{P}_2' = 0$	The point $\bar{\bar{y}}'''$ is on the line $\bar{\bar{y}} \ \bar{\bar{y}}'$. The point $\bar{\bar{y}}^{(4)}$ is on the line $\bar{\bar{y}}' \ \bar{\bar{y}}''$.
If $\bar{P}_3' = 0$	The point $\bar{\bar{y}}'''$ is on the line $\bar{\bar{y}} \ \bar{\bar{y}}'$. The point $\bar{\bar{y}}^{(4)}$ is on the line $\bar{\bar{y}}' \ \bar{\bar{y}}''$.
If $3\bar{P}_2' + \bar{P}_3 = 0$	The point $\bar{\bar{y}}'''$ is on the line $\bar{\bar{y}} \ \bar{\bar{y}}'$. The point $\bar{\bar{y}}^{(4)} \equiv \bar{\bar{y}}''$. The curve generated by the point $\bar{\bar{y}}'$ is a straight line.

The Family of Eight Pointic Cubics. The Nine-Pointic Cubic. The Halphen Point.

In the theory of plane cubic curves, it is shown that if we have two eight pointic cubics $u = 0$, and $v = 0$ then $u + kv = 0$ represents a family of eight pointic cubics. Among the curves of this family there is a nine-pointic cubic, that is, the osculating cubic of the curve generated by $\bar{\bar{y}}$.

If we substitute the value of η from equation (23) in the equations $u = 0$, $v = 0$ we have

$$u = 5\xi^3 + 15\bar{P}_3\xi\eta^2 + 2\eta^3 - 10\xi\eta$$

$$= 3/112\ \bar{P}_3\xi^8 + \frac{1}{2240}\left[5\bar{P}_3 - 1\right]\xi^9 + ---,$$

$$v = 45\bar{P}_3\xi^2\eta - 7\xi\eta^2 + 135\bar{P}_3^2\eta^3$$

$$- 35\xi^2 - 195\bar{P}_3\eta^2 + 70\eta$$

$$= \frac{1}{320}\left[29 - 5\bar{P}_3\right]\xi^8 + \frac{1}{4032}\left[108\bar{P}_3^2 - 7\bar{P}_3'\right]\xi^9 + ---$$

From these equations the elimination of ξ^8 gives us

$$\frac{3}{112}\bar{P}_3\ v - \frac{1}{320}\left[29 - 5\bar{P}_3\right]u$$

$$= \frac{10800\,\bar{P}_3^{\,3} - 700\,\bar{P}_3\,\bar{P}_3' - 3150\,\bar{P}_3' + 525\,\bar{P}_3'^{\,2} + 609}{15052800}\quad \rho_9,$$

where

$$\frac{3}{112}\,\bar{P}_3\,v - \frac{1}{320}\left[29 - 5\bar{P}_3'\right]u = 0$$

is the osculating nine-pointic cubic. Upon substituting the values of u and v and changing to homogeneous coordinates, we have

$$(38)\qquad \begin{aligned}&\left[1015 - 175\,\bar{P}_3'\right] x_2^3 - 2700\,\bar{P}_3^{\,2} x_2^2\, x_3\\ &+ \left[3465\,\bar{P}_3 - 525\,\bar{P}_3\,\bar{P}_3'\right] x_2\, x_3^2\\ &+ \left[406 - 70\,\bar{P}_3' - 8100\,\bar{P}_3^{\,3}\right] x_3^3\\ &+ 2100\,\bar{P}_3\, x_1 x_2^2 + \left[350\,\bar{P}_3' - 2030\right] x_1 x_2\, x_3\\ &+ 11700\,\bar{P}_3^{\,2}\, x_1\, x_3^2 - 4200\,\bar{P}_3\, x_1^2\, x_3 = 0\end{aligned}$$

The eight pointic cubics have a ninth point in common which we shall call the Halphen point of $\bar{\bar{y}}$. We shall proceed to find its expression.

From equations (29) and (30) we have

$$(39)\qquad u = 5(x_2^2 - 2x_1 x_3)\, x_3 + x_3^2\,(15\bar{P}_3\, x_2 + 2x_3) = 0$$

$$(40)\qquad \begin{aligned}v = {}&5(x_2^2 - 2x_1 x_3)\,(9\bar{P}_3 x_3 - 7x_1)\\ &+ 15\bar{P}_3 x_3^2\,(9\bar{P}_3 x_3 - 7x_1) - 7x_2\, x_3^2 = 0\end{aligned}$$

We find from (39)

$$5\,(x_2^{2} - 2x_1\, x_3) = -\frac{x_3^{2}}{x_2}\,(15\,\bar{P}_2\, x_2 + 2x_3)\,,$$

and $$x_1 = \frac{5\, x_2^{3} + 15\,\bar{P}_2\, x_2\, x_3^{2} + 2\, x_3^{3}}{10\, x_2\, x_3}$$

These give further

$$9\,\bar{P}_2\, x_3 - 7x_1 = -\frac{15\,\bar{P}_2\, x_2 x_3^{2} + 35\, x_2^{3} + 14\, x_3^{3}}{10\, x_2\, x_3}$$

Whence, substituting in (40)

$$\frac{15\,\bar{P}_2\, x_2\, x_3^{4} + 14\, x_3^{5}}{5x_2^{2}} = 0$$

The solution $x_3 = 0$ gives $x_2 = 0$, that is, the point $\bar{\bar{y}}$. The other solution gives

$$x_2 = w\, 14$$

$$x_3 = w\, 15\, P_2$$

where w is a proportionality factor. When we substitute in the expression for x_1 we have

$$x_1 = w\,\frac{686 + 2700\,\bar{P}_2^{3}}{105\,\bar{P}_2}$$

Let $w = 105\,\bar{P}_2$ then

$$x_1 = 686 + 2700\,\bar{P}_2^{3}$$

$$x_2 = 1470\,\bar{P}_2$$

$$x_3 = 1575\,\bar{P}_2^{2}$$

or

(41) $$h = [686 + 2700\,\bar{P}_2^3]\,\bar{\bar{y}} + 1470\,\bar{P}_2\,\bar{\bar{y}}' + 1575\,\bar{P}_2^2\,\bar{\bar{y}}'',$$

which is the expression for the Halphen point.

The polar line of the Halphen point with respect to the osculating conic has the equation

(42) $$1575\,\bar{P}_2^2\,x_1 - 1470\,\bar{P}_2 x_2 + (686 - 2025\,\bar{P}_2^3)\,x_3 = 0$$

If $\bar{P}_2 = \left(-\frac{686}{2700}\right)^{1/3}$ the Halphen point is on the line joining $\bar{\bar{y}}'$, $\bar{\bar{y}}''$ and has the coordinates

$$\left(0,\ 1470\left[-\frac{686}{2700}\right]^{1/3},\ 1575\left[-\frac{686}{2700}\right]^{\frac{2}{3}}\right)$$

The polar line of the Halphen point intersects $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ in the points

$$(0,\ 686 - 2025\,\bar{P}_2^3,\ 1470\,\bar{P}_2),$$

$$(2025\,\bar{P}_2^3 - 686,\ 0,\ 1575,\ \bar{P}_2^2),$$

$$(14\,\bar{P}_2,\ 15,\ 0)$$

respectively. It intersects the line $x_1 = 3\,\bar{P}_2\,x_3$ in the point

$$(2205\,\bar{P}_2^2,\ 343 + 1350\,\bar{P}_2^3,\ 735\,\bar{P}_2)$$

and the line $2x_1 = 3\,\bar{P}_2\,x_3$ in the point

$$(4410\,\bar{P}_2^2,\ 675\,\bar{P}_2^3 + 1372,\ 2940\,\bar{P}_2)$$

In general, two curves of degree n and m intersect in n m points. Therefore, in general, each cubic of our family of cubics and our osculating conic intersect in six points.

If we solve the equation of our osculating conic with the equation of our family of cubics we find the solutions

(43) $(1,0,0)$, $(4 + 147 \bar{P}_2 k^2 , 28 k, 98 k^2)$.

When $k = 0$ the solutions determine the same point $(1,0,0)$. When $k = \dfrac{2 i \sqrt{3 \bar{P}_2}}{21 \bar{P}_2}$, the sixth point of intersection is on the line $\bar{\bar{y}}' \; \bar{\bar{y}}''$, that is, $(0, i\sqrt{3 \bar{P}_2} , -8)$.

IX.

Particular Eight-Pointic Cubics.

We have, the family of eight-pointic cubics $u + kv = 0$ or

$$(44) \qquad (5x_2^3 + 15\bar{P}_2 x_2 x_3^2 + 2x_3^3 - 10x_1 x_2 x_3)$$

$$+ k\,(45\bar{P}_2 x_2^2 x_3 - 7x_2 x_3^2 + 135\bar{P}_2^2 x_3^3$$

$$- 35x_1 x_2^2 - 195\bar{P}_2 x_1 x_3^2 + 70x_1^2 x_3) = 0$$

If $k = -\dfrac{2}{135\bar{P}_2^2}$ we obtain

$$675\bar{P}_2^2 x_2^3 + (2025\bar{P}_2^3 + 14)x_2 x_3^2 - 1350\bar{P}_2^2 x_1 x_2 x_3$$

$$- 90\bar{P}_2 x_2^2 x_3 + 70x_1 x_2^2 + 390\bar{P}_2 x_1 x_3^2$$

$$- 140x_1^2 x_3 = 0,$$

the equation of the eight pointic cubic through the point $(0,0,1)$.

If $\bar{P}_2^3 = -\dfrac{14}{2025}$ then the tangent at the point $(0,0,1)$ is the line $x_1 = 0$.

If $\bar{P}_2 = 0$, the tangent at $(0,0,1)$ is the line $x_2 = 0$.

If $k = -\dfrac{1}{90\bar{P}_2^2}$ we obtain

$$450\,\bar{P}_2^{2}\,x_2^{3} + 1350\,\bar{P}_2^{3}\,x_2\,x_3^{2} + 45\,\bar{P}_2^{2}\,x_3^{3}$$

$$-\,900\,\bar{P}_2^{2}\,x_1 x_2 x_3 - 45\,\bar{P}_2\,x_2^{2}\,x_3 + 7\,x_2\,x_3^{2}$$

$$+\,35\,x_1\,x_2^{2} + 195\,\bar{P}_2\,x_1\,x_3^{2} - 70\,x_1^{2}x_3 = 0$$

the equation of the eight-pointic cubic through the point $(3\bar{P}_2\ , 0, 1)$ with

$$-\,225\,\bar{P}_2\,x_1 + (7 - 1350\,\bar{P}_2^{3})\,x_2 + 675\,\bar{P}_2^{2}\,x_3 = 0,$$

the tangent at the point $(3\bar{P}_2\ , 0, 1)$. Now, if

$\bar{P}_2 = 0$, $x_2 = 0$ is the tangent at $(0,0,1)$ and if $\bar{P}_2 = (7/1350)^{1/3}$, $-\,225\,[7/1350]^{1/3}x_1 + 675\,[7/1350]^{2/3}$

$$x_3 = 0$$

is the tangent at the point $(3\,(7/1350)^{1/3}, 0,1)$.

In our family of cubics $u + kv = 0$ there will be some which are tangent to the line $x_1 = 0$ of our triangle of reference. From (44) we find that the equation of the tangent to the cubic (44) at a point, (o, b_1, c_1) on the line $x_1 = 0$ is

$$[-10\,b_1\,c_1 - k\,(35\,b_1^{2} + 195\,\bar{P}_2\,c_1^{2})]\,x_1$$

$$+\,[15\,b_1^{2} + 15\,\bar{P}_2\,c_1^{2} + k\,(90\,\bar{P}_2\,b_1 c_1 - 7\,c_1^{2})]\,x_2$$

$$+\left[30\,\overline{P_2}\,b_{,}c_{,} + 6\,c_{,}^2 + k\,(45\,\overline{P_2}\,b_{,}^2 - 14\,b_{,}c_{,} + 405\,\overline{P_2}^2\,c_{,}^2)\right]x_3 = 0.$$

In order that the tangent be the line $x_{,} = 0$ we must have

$$15\,b_{,}^2 + 15\,\overline{P_2}\,c_{,}^2 + k\,(90\,\overline{F_2}\,b_{,}c_{,} - 7\,c_{,}^2) = 0$$

and

$$30\,\overline{P_2}\,b_{,}c_{,} + 6\,c_{,}^2 + k\,(45\,\overline{P_2}\,b_{,}^2 - 14\,b_{,}c_{,} + 405\,\overline{P_2}^2\,c_{,}^2) = 0.$$

Sylvester's dialytic method of elimination gives us an equation of fourth degree in k, therefore, there will be four cubics of our family tangent to the line $x_{,} = 0$ at a point $(o, b_{,}, c_{,})$. If two pairs of the values for k are equal, there will be two cubics.

The eight pointic cubic $v = 0$ passes through the point (0, 1, 0) is tangent to the line $x_3 = 0$ at the point (1,0,0), and intersects the line $x_2 = 0$ at the points (1,0,0), $(9\,\overline{P_2}, 0, 7)$ and $(3\,\overline{P_2}, 0, 2)$. The line $7x = 9\,\overline{P_2}\,x_3$ is tangent to this cubic at the point (0,1,0).